P.G. Semester-IV Examination, 2023
MATHEMATICS

Course ID : 42152 **Course Code : MATH402C**

Course Title : Graph Theory & Field Theory

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

GROUP-A

(Graph Theory)

Answer any **two** from the following three questions: $8\times 2=16$

1. a) Prove that a graph is bipartite if and only if it does not contain any cycle of odd length.

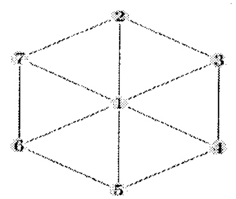
   b) Draw a directed graph whose adjacency matrix is given by

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

   c) Does there exist a simple graph with degree sequence 5,5,4,2,2,2? Justify your answer. 5+1+2

2. a) Define the center of a graph. Find the number of center(s) for a tree.

   b) Find the Chromatic polynomial for the following graph:



(1+3)+4

3. a) Prove that a graph G has an Euler trail if and only if G is a connected graph having exactly two vertices of odd degrees.

   b) Let G be a simple planar graph with n vertices, e edges and k components. If each component of G contains at least 3 vertices, then prove that $e \leq 3n-6k$. Hence check the planarity of the graph K_5. 3+(4+1)

GROUP-B

(Field Theory)

Answer any **three** from the following five questions.

$8\times3=24$

4. a) Let F be an extension field of K and $u \in F$ be algebraic over K. Find $[K(u): K]$ by obtaining a basis.

   b) Find the dimension of the field extension $\mathbb{Q}(i, w, \sqrt{3}, \sqrt{12})$ over $\mathbb{Q}$ with justification.

   c) Are the fields $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(e)$ isomorphic to each other? Justify your answer. 3+3+2

5. a) Let F over K be a field extension and X be a set of algebraic elements of F over K. Prove that $K(X)$ is algebraic over K. Hence give an example of an infinite dimensional algebraic extension.

   b) Let K be a field and $v=\dfrac{x^5}{x-4}\in K(x)$. Show that $K(x)$ is a simple algebraic extension over $K(v)$. Also, find $[K(x):K(v)]$. (3+1)+(2+2)

6. a) Find the Galois group of the field extension $\mathbb{Q}(2^{\frac{1}{3}}, 2^{\frac{2}{3}})$ over $\mathbb{Q}$.

   b) State Fundamental theorem for Galois Theory of finite dimensional field extension.

   c) If G is a finite group, show that there exists a Galois field extension with the Galois group isomorphic to G. 3+2+3

7. a) Define a separable field extension. Give an example of an algebraic extension which is not separable.

   b) Define a cyclic field extension.

   c) Find the Galois group of the polynomial $x^2-4x+2\in\mathbb{Q}[x]$. (1+3)+1+3

8. a) If F is a finite field, then prove that the group $(F^*,.)$ is cyclic.

   b) Construct fields with 9 and 12 elements, respectively.

   c) Find the splitting field of the polynomial $(x^3-5)(x^2-16)(x^2-5)$ over $\mathbb{Q}$. 3+(2+1)+2

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